Integral formulae

1 Arithmetic formulae

Type - I — Normal

$y = f(x)$

y ⇓ dept variable

x ⇓ Indept variable

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \; ; \; n \neq -1$

(ii) $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a \cdot (n+1)} + c \; ; \; n \neq -1$

(iii) $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + c$

(iv) $\int \sqrt{ax+b} \, dx = \frac{2}{3} \frac{(ax+b)^{3/2}}{a} + c$

$\left(\because \int \sqrt{x} \, dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2} \right)$

(v) $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$

(i.e.) $\int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$

(vi) $\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a} + c$

Methods of Int.

(i) Substitution method

(i) $\left.\begin{matrix} Nr \text{ or} \\ Dr \end{matrix}\right\}$ ⇒ within in bracket like $(ax+b)$

let $\boxed{\begin{matrix} (\ ) = t \\ \text{complet} = t \end{matrix}}$

(ii) $\dfrac{Nr}{dr} =$

$\dfrac{d}{dx}(dr) = Nr.$ ⇒ special case.

$\int \dfrac{dx}{x}$

II - Trigonometric formulae:

1. Functions | Reciprocals

$\sin x = \frac{1}{\operatorname{cosec} x} \Rightarrow \operatorname{cosec} x = \frac{1}{\sin x}$

$\cos x = \frac{1}{\sec x} \Rightarrow \sec x = 1/\cos x$

$\tan x = \frac{1}{\cot x} \Rightarrow \cot x = 1/\tan x.$

$\boxed{\frac{1}{\sin x} \neq \sin^{-1} x}$ (Inverse not exist.)

Identity

(+) $\sin x$ | $\operatorname{cosec} x$

$\cos x$ (-ve) | $\sec x$ (+ve)

$\tan x$ | $\cot x$

(1) $\boxed{\sin^2 x + \cos^2 x = 1}$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

(2) $\boxed{\sec^2 x - \tan^2 x = 1}$

$\sec^2 x = 1 + \tan^2 x$

$\tan^2 x = \sec^2 x - 1$

(3) $\boxed{\operatorname{cosec}^2 x - \cot^2 x = 1}$

$\operatorname{cosec}^2 x = 1 + \cot^2 x$

$\cot^2 x = \operatorname{cosec}^2 x - 1$

4 $\boxed{\sin 2x = 2\sin x \cos x}$

8 $\sin x = 2\sin x/2 \cos x/2$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

5 $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

6 $$\boxed{\sin^2 x = \frac{1-\cos 2x}{2}}$$

$2\sin^2 x = 1 - \cos 2x$

$\cos 2x = 1 - 2\sin^2 x$

$\cos x = 1 - 2\sin^2 x/2$

7 $$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$2\cos^2 x = 1 + \cos 2x$

$\cos 2x = 2\cos^2 x - 1$

$\cos x = 2\cos^2 x/2 - 1$

(8) $$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

(9) $$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\boxed{\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]}$$

(10) $$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\boxed{\cos^3 x = \frac{1}{4}\left[\cos 3x + 3\cos x\right]}$$

(11) $$\boxed{\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}.}$$

(12) $$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

(13) $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

14 $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B.$

15 Let $\boxed{A > B}$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right].$

Type - I $\Rightarrow$ Normal

linear power $\begin{cases} \to \sin x \\ \to \cos x \end{cases}$ $\quad$ $\left.\begin{matrix} \operatorname{cosec} x \\ \sec x \end{matrix}\right] \Rightarrow$ power '2'

$\underbrace{\tan x \qquad \cot x}$

Identity.

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(1) $\int \sin x \, dx = -\cos x + c$

i.e. $\int \sin(ax+b) = -\frac{1}{a}\cos x + c$

(2) $\int \cos x \, dx = \sin x + c$

i.e. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(3) $\int \sec^2 x \, dx = \tan x + c$

(4) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

(5) $\int \sec x \tan x \, dx = \sec x + c$

(6) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

Type-II → Special

(7) $\int \tan x \, dx = \log|\sec x| + c$

$\int \tan(ax+b) \, dx = \frac{1}{a} \log|\sec(ax+b)| + c$

(8) $\int \cot x \, dx = \log|\sin x| + c$

$\int \cot(ax+b) \, dx = \frac{1}{a} \log|\sin(ax+b)| + c$

(9) $\int \sec x \, dx = \log|\sec x + \tan x| + c$

$\int \sec(ax+b) \, dx = \frac{1}{a} \log|\sec(ax+b) + \tan(ax+b)| + c$

(10) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

$\int \operatorname{cosec}(ax+b) \, dx = \frac{1}{a} \log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + c$

Type - II

(1) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$ by $\sec^2 x - \tan^2 x =$ $\boxed{\tan^2 x = \sec^2 x - 1}$

$$\boxed{\int \tan^2 x\,dx = \tan x - x + c}$$

(2) $\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)\,dx$

$= -\cot x - x + c$

(3) $\int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$\because \sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\left(\int dx - \int \cos 2x\,dx\right)$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

(4) $\int \cos^2 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right) dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

Type – III

(1) $\int \sin A \cos \cancel{A}\, dx$

(ie) $\int \underline{\sin x}\, \underline{\cos x}\, dx = \frac{1}{2}\int 2\sin x \cos x\, dx$

same same

$= \frac{1}{2}\int \sin 2x\, dx$

$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$

$= -\frac{1}{4}\cos 2x + c$

(2) $\int \sin A \cos B\, dx = \frac{1}{2}\int (\sin(A+B) + \sin(A-B))\, dx$ if $\boxed{A > B}$

IIIly we can apply all the 4' formulae

Special formulae

(1) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

(2) $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

(3) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1} x/a + c$$

$$\int \frac{dx}{x^2+1} = \tan^{-1} x + c$$

(4) $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} x/a + c$$

(5) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

(6) $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

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(4)' $$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$

7 $\int \frac{dx}{ax^2+bx+c}$ (or) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

let $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+c/a\right)$

Completing square method $\Big\}$ $= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{a^2}\right]$

$= a\left(t^2+\left(\frac{c}{a}-\frac{b^2}{a^2}\right)\right)$

$\because x+b/2a = t.$

8 $\int \frac{px+q}{ax^2+bx+c}\,dx$ & $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

let $px+q = A\,\frac{d}{dx}(ax^2+bx+c)+B.$

(9) **Type 1**: $\int \frac{dx}{ax^2+bx+c}$ (or)

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Step: 1

$$ax^2+bx+c$$

$$= a\left[x^2+\frac{b}{a}x+c/a\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

$\Downarrow$ Const

$$x+b/2a=t$$

$$dx=dt$$

(9) Type-II $\int \frac{px+q}{ax^2+bx+c}$ $\Rightarrow \frac{\text{linear pow}}{\text{Q. pow}}$

(or)

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

Consider $px+q = A\frac{d}{dx}(ax^2+bx+c) + B.$

Equating like co-efficient

Integration by partial fractions

* A rational fraction is defined as the ratio of 2 - polynomials

(ie) $\frac{P(x)}{Q(x)}$

wr $P(x)$ & $Q(x)$ = Polynomials in x & $Q(x) \neq 0$.

* If the $\boxed{\text{degree of } P(x) < \deg Q(x)}$ then the rational fraction is called proper.

otherwise it is called improper

* If the rational fraction is improper then reduce the function into a proper rational function.

Form of the rational functions	Form of the partial fractions.
1) $\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$
2) $\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
3) $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$
4) $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
5) $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)} + \dfrac{Bx+C}{x^2+bx+c}$

* Wr x^2+bx+c can't be factorised further.

* A, B, C are the real numbers to be determined suitably.

Integration by parts.

If 'u' & 'v' are any two-diff! func of a variable 'x', then

(1) $$\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\left(\int v\,dx\right)]\,dx + c$$

(or) $$\int u\,dv = \overline{\overline{uv - \int v\,du}}.$$

(2) **Bernoullis formulae**

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + \dots$$

u' = differentiation.

v_1 = Integration.

ILATE (first in order).

I - Inverse trigonometric function

L - Logarithmic

A - Arithmetic

T - Trigonometric

E - exponentia fun

Definite integral as the limit of a sum:

Let $y = f(x)$ be a non-negative continuous function defined on the closed interval $[a, b]$

then $$\int_a^b f(x)\,dx = \underset{h \to 0}{Lt}\ h[f(a) + f(a+h) + \cdots + f(a + (\overline{n-1})h$$

Wr $\boxed{nh = b - a}$ (or) $\boxed{h = \dfrac{b-a}{n}}$

Some useful formulae:

1) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

2) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Integrals of some more types:-

(1) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log[x+\sqrt{x^2-a^2}] + C$

(2) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c$

(3) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

x (4) Integral of the type

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx$$

$$(px+q) = A\frac{d}{dx}(ax^2+bx+c) + B. \quad \text{(1)}$$

$$(px+q) = A(2ax+b) + B.$$

comparing ~~eq~~ like co-efficients.

$$2aA = p \quad ; \quad Ab + B = q.$$

$$\boxed{A = \frac{p}{2a}} \qquad \boxed{B = q - Ab.}$$

$$\boxed{B = q - \frac{b \cdot p}{2a}}$$

Substing the values in (1), then $\int^{te}$, find the answer

Integration by Parts:

If u and v are any two differential functions of a variable x, then

$$\int uv\, dx = u\int v\, dx - \int \left[\frac{du}{dx}\int v\, dx\right] + c$$

(or)

$$\boxed{\int u\, dv = uv - \int v\, du.}$$

Remark

We shall choose the first function by the word $\boxed{\text{ILATE}}$

I - Inverse trigonometric fun

L - Logarithmic fun

A - Algebric fun

T - Trigonometric "

E - Exponential "

Fundamental theorem of Calculus:-

1) **First fundamental theorem of integral calculus:-**

Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $\boxed{A'(x) = f(x)}$ for all $x \in [a, b]$

2) **Second fundamental theorem of integral calculus:-**

Let f be a continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f.

Then $\int_a^b f(x)\,dx = \Big[f(x) \Big]_a^b = F(b) - F(a).$

Some Properties of Definite integrals

Prop: 1 : $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy.$

Prop: 2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

prop: 3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Wr $a < c < b$

prop: 4 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

Prop: 5 $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

prop: 6 $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Prop: 7 $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

Prop: 8

(i) $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$ if $f(-x) = f(x)$

(ii) $$\int_{-a}^{a} f(x)\,dx = 0$$ if $f(-x) = -f(x)$.