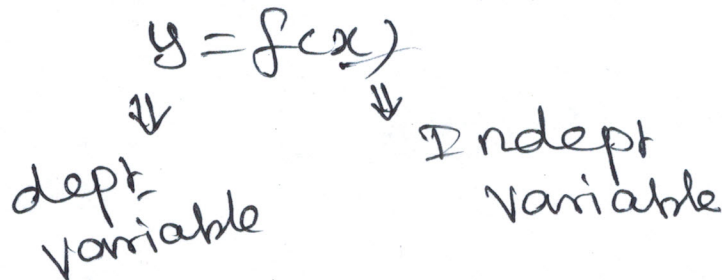


# Integral formulae

## Arithmetic formulae

### Type - I - Normal



(i)  $\int x^n dx = \frac{x^{n+1}}{n+1} + c ; n \neq -1$

(ii)  $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \cdot (n+1)} + c ; n \neq -1$

(iii)  $\int \sqrt{x} dx = \frac{2}{3} x^{3/2} + c$

(iv)  $\int \sqrt{ax+b} dx = \frac{2}{3} \frac{(ax+b)^{3/2}}{a} + c$

$(\because \int \sqrt{x} dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1}$   
 $= \frac{x^{3/2}}{3/2}$   
 $= \frac{2}{3} x^{3/2})$

(v)  $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$

(vi)  $\int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1}$

$= \frac{x^{1/2}}{1/2} = 2\sqrt{x}$

(vii)  $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a} \sqrt{ax+b} + c$

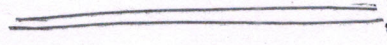
# Methods of Int.

## (i) Substitution method

(i)  $\frac{Nr}{Dr}$  }  $\Rightarrow$  within in bracket like  $(ax+b)$

(ii)  $\frac{Nr}{dr} =$   $( ) = t$   
 $complet = t$

$\frac{d}{dx}(dx) = Nr. \Rightarrow$  special case  
 $\int \frac{dx}{x}$



# II - Trigonometric formulae

Functions

$$\sin x = \frac{1}{\operatorname{cosec} x}$$

$$\cos x = \frac{1}{\sec x}$$

$$\tan x = \frac{1}{\cot x}$$

Reciprocals

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

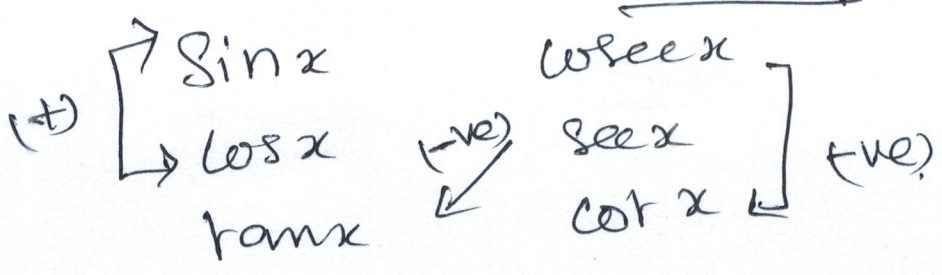
$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

$$\boxed{\sin x \neq \sin^{-1} x}$$

(Inverse not exist.)

Identity



$$\textcircled{1} \quad \boxed{\sin^2 x + \cos^2 x = 1}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\textcircled{2} \quad \boxed{\sec^2 x - \tan^2 x = 1}$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\textcircled{3} \quad \boxed{\operatorname{cosec}^2 x - \cot^2 x = 1}$$

$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

④

$$\sin 2x = 2 \sin x \cos x$$

⑤

$$\sin x = 2 \sin x/2 \cos x/2$$

$$\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$$

⑥

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

⑦

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\cos x = 1 - 2 \sin^2 x/2$$

⑧

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos x = 2 \cos^2 x/2 - 1$$

$$\textcircled{8} \quad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\textcircled{9} \quad \sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\boxed{\sin^3 x = \frac{1}{4} [3 \sin x - \sin 3x]}$$

$$\textcircled{10} \quad \cos 3x = 4 \cos^3 x - 3 \cos x$$

$$\boxed{\cos^3 x = \frac{1}{4} [\cos 3x + 3 \cos x]}$$

$$\textcircled{11} \quad \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$\textcircled{12} \quad \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\textcircled{13} \quad \tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

14  $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

Let

15  $\boxed{A > B}$

$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$

$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$

$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

Type - I  $\Rightarrow$  Normal  
 $\left. \begin{matrix} \sec x \\ \csc x \end{matrix} \right\} \Rightarrow$  power '2'

Linear power  $\left\{ \begin{matrix} \rightarrow \sin x \\ \rightarrow \cos x \end{matrix} \right.$

$\left. \begin{matrix} \tan x & \cot x \end{matrix} \right\} \downarrow$   
Identity.

- ①  $\int \sin x \, dx = -\cos x + C$   
 (i)  $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
- ②  $\int \cos x \, dx = \sin x + C$   
 (i)  $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
- ③  $\int \sec^2 x \, dx = \tan x + C$
- ④  $\int \csc^2 x \, dx = -\cot x + C$
- ⑤  $\int \sec x \tan x \, dx = \sec x + C$
- ⑥  $\int \csc x \cot x \, dx = -\csc x + C$



7

Type-II → special

$$\int \tan x \, dx = \log |\sec x| + c$$

$$\int \tan(ax+b) \, dx = \frac{1}{a} \log |\sec(ax+b)| + c$$

8

$$\int \cot x \, dx = \log |\sin x| + c$$

$$\int \cot(ax+b) \, dx = \frac{1}{a} \log |\sin(ax+b)| + c$$

9

$$\int \sec x \, dx = \log |\sec x + \tan x| + c$$

$$\int \sec(ax+b) \, dx = \frac{1}{a} \log |\sec(ax+b) + \tan(ax+b)| + c$$

10

$$\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$$

$$\int \operatorname{cosec}(ax+b) \, dx = \frac{1}{a} \log \left| \operatorname{cosec}(ax+b) - \cot(ax+b) \right| + c$$





Type-II

(9)

$$\textcircled{1} \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$

by  $\sec^2 x - \tan^2 x = 1$

$$\boxed{\int \tan^2 x \, dx = \tan x - x + c}$$

$$\boxed{\tan^2 x = \sec^2 x - 1}$$

$$\textcircled{2} \int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx$$

$$= -\cot x - x + c$$



$$\textcircled{3} \int \sin^2 x \, dx = \int \left( \frac{1 - \cos 2x}{2} \right) dx$$

$$\therefore \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left( \int dx - \int \cos 2x \, dx \right)$$

$$= \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right] + c$$

$$\textcircled{4} \int \cos^2 x \, dx = \int \left( \frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right] + c$$



## Type - III

(10)

$$\textcircled{1} \int \sin A \cos A dx$$

$$\begin{aligned} \text{(a)} \int \sin x \cos x dx &= \frac{1}{2} \int 2 \sin x \cos x dx \\ &\quad \begin{array}{c} \downarrow \quad \downarrow \\ \text{same} \quad \text{same} \end{array} \\ &= \frac{1}{2} \int \sin 2x dx \\ &= \frac{1}{2} \left( -\frac{\cos 2x}{2} \right) + C \\ &= -\frac{1}{4} \cos 2x + C \end{aligned}$$

$$\textcircled{2} \int \sin A \cos B dx \quad \leftarrow \text{if } \boxed{A > B}$$
$$= \frac{1}{2} \int (\sin(A+B) + \sin(A-B)) dx$$

Why we can apply all the 4' formulae

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## Special formulae

①  $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

②  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

③  $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} x/a + c$

$\int \frac{dx}{x^2 + 1} = \tan^{-1} x + c$

④  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} x/a + c$

⑤  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log | x + \sqrt{x^2 - a^2} | + c$

⑥  $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log | x + \sqrt{x^2 + a^2} | + c$

④'  $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + c$

7

$$\int \frac{dx}{ax^2+bx+c}$$

(or)  $\int \frac{dx}{\sqrt{ax^2+bx+c}}$  (12)

let  $ax^2+bx+c = a(x^2 + \frac{b}{a}x + \frac{c}{a})$

Completing  
square  
method

$$= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right]$$

$$= a \left[ t^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

$\therefore x + \frac{b}{2a} = t$

8

$$\int \frac{px+q}{ax^2+bx+c} dx$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

let  $px+q = A \frac{d}{dx}(ax^2+bx+c) + B$

④ Type 1:  $\int \frac{dx}{ax^2+bx+c}$  (or)

$\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Step: 1

$ax^2+bx+c$

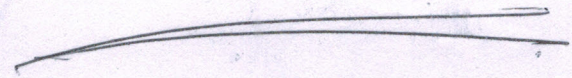
$= a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right]$

$= a \left[ \left( x + \frac{b}{2a} \right)^2 + \left( \frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$

↓ Const

$x + \frac{b}{2a} = t$

$dx = dt$



9

Type-II

$$\int \frac{px+q}{ax^2+bx+c} \Rightarrow \frac{\text{linear}}{\text{quad.}} \Rightarrow \frac{\text{linear}}{\text{quad.}}$$

(or)

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

Consider  $px+q = A \frac{d}{dx}(ax^2+bx+c) + B$

Equating like coefficients

## Integration by partial fractions

(13)

\* A rational fraction is defined as the ratio of 2 - polynomials

$$\text{(ie) } \frac{P(x)}{Q(x)}$$

Wt  $P(x) \neq Q(x) = \text{Polynomials}$   
in  $x$  s.  $Q(x) \neq 0$ .

\* If the degree of  $P(x) < \text{deg } Q(x)$

then the rational fraction is called proper.

otherwise it is called improper.

\* If the rational fraction is improper then reduce the function into a proper rational function.

---

Form of the rational functions	Form of the partial fractions.
1) $\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2) $\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4) $\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

or \*  $x^2+bx+c$  can't be factorised further.

\* A, B, C are the real numbers to be determined suitably.

Form of the rational function



# Integration by parts.

If 'u' & 'v' are any two-diff<sub>l</sub> functions of a variable 'x', then

$$\int uv dx = u \int v dx - \int \frac{du}{dx} (\int v dx) dx + c$$

(or)  $\int u dx = \underline{\underline{uv}} - \int v du.$

## Bernoulli's formulae

$$\int uv dx = uv_1 - u'v_2 + u''v_3 + \dots$$

u' = differentiation.

v<sub>1</sub> = Integration.

## ILATE (first in order).

I - Inverse trigonometric function

L - Logarithmic

A - Arithmetic

T - Trigonometric

E - exponential fun

(b)

Definite integral as the limit of a  
sum;

Let  $y = f(x)$  be a non-negative continuous function defined on the closed interval  $[a, b]$

$$\text{Then } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} h [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

$$\text{Wr } \boxed{nh = b-a} \quad (\text{or}) \quad \boxed{h = \frac{b-a}{n}}$$

Some useful formulae;

$$1) \quad 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$2) \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$3) \quad 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

---

## Integrals of some more types:-

$$(1) \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$$

$$(2) \int \sqrt{x^2 + a^2} dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + C$$

$$(3) \int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

x (4) Integral of the type

$$\int (px + q) \sqrt{ax^2 + bx + c} dx$$

$$(px + q) = A \frac{d}{dx} (ax^2 + bx + c) + B$$

$$(px + q) = A(2ax + b) + B$$

Comparing ~~eq.~~ like co-efficients.

$$2aA = p \quad ; \quad Ab + B = q$$

$$\boxed{A = \frac{p}{2a}}$$

$$\boxed{B = q - Ab}$$

$$\boxed{B = q - \frac{b \cdot p}{2a}}$$

Substituting the values

in (1), then  $\int$  to find the answer

# Integration by Parts:

If  $u$  and  $v$  are any two differential functions of a variable  $x$ , then

$$\int uv \, dx = u \int v \, dx - \int \frac{du}{dx} \int v \, dx + c$$

(or)

$$\int u \, dv = uv - \int v \, du.$$

## Remark

We shall choose the first function by the word I.L.A.T.E

- I - Inverse trigonometric fun
- L - Logarithmic fun
- A - Algebraic fun
- T - Trigonometric "
- E - Exponential "



# Fundamental Theorem of Calculus:-

## 1) First Fundamental Theorem of Integral Calculus:-

Let  $f$  be a continuous function on the closed interval  $[a, b]$  and let  $A(x)$  be the area function.

Then  $A'(x) = f(x)$  for all  $x \in [a, b]$

## 2) Second Fundamental Theorem of Integral Calculus:-

Let  $f$  be a continuous function defined on the closed interval  $[a, b]$  and  $F$  be an antiderivative of  $f$ .

Then  $\int_a^b f(x) dx = [f(x)]_a^b = F(b) - F(a)$ .



# Some Properties of Definite Integrals

Prop: 1 :  $\int_a^b f(x) dx = \int_a^b f(y) dy.$

Prop: 2  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

Prop: 3  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$   
Wr  $a < c < b$

Prop: 4  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$

Prop: 5  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

Prop: 6  $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$

Prop: 7  $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$

if  $f(2a-x) = f(x)$   
 $= 0$  if  $f(2a-x) = -f(x)$

Prop: 8

$$(i) \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$$\text{if } f(-x) = f(x)$$

$$(ii) \int_{-a}^a f(x) dx = 0 \quad \text{if } f(-x) = -f(x).$$

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